Entrepreneurial Business and Economics Review

2023, Vol. 11, No. 3

OPEN ACCESS

10.15678/EBER.2023.110312

Using trigonometric seasonal models in forecasting the size of withdrawals from automated teller machines

Henryk Gurgul, Łukasz Lach, Marcin Suder, Karol Szpyt

A B S T R A C T

Objective: The study focused on verifying the impact of the calendar and seasonal effects on the accuracy of forecasts of cash withdrawals from automated teller machines (ATMs). In this article, we investigated a possible use of the so-called trigonometric seasonality, the Box-Cox transformation, ARMA errors, trend, and seasonal components (TBATS) models to forecast withdrawals from ATMs. In practice, the SARIMA model is widely used as a forecasting tool. However, the major limitation of SARIMA models is that it allows just one single seasonality pattern to be taken into account, e.g., weekly seasonality. At the same time, cash withdrawals from ATMs display overlapping multi-seasonality. Therefore, the goal of this article is to compare the SARIMA model with the TBATS model, both in basic forms and forms extended with event-specific dummies.

Research Design & Methods: Empirical research was conducted by means of fitting SARIMA and TBATS models to daily time series of withdrawals from 74 ATMs managed by one of the largest ATM operators in Poland. The dataset covered the period of 2017-2019.

Findings: Forecasting levels of cash withdrawals plays a crucial role in the management of ATM networks, both in the case of a single ATM as well as the whole network. Prediction accuracy has a direct impact on the operational costs of the network. These costs result from activities such as freezing cash in an ATM, preparing it, and transporting it to an ATM. Therefore, the choice of a proper forecast model is of special importance. According to statistical evidence in our study, the basic TBATS model gives more accurate forecasts than the basic SARIMA model widely used in practice.

Implications & Recommendations: The multi-seasonality of ATM withdrawals means that it is necessary to use techniques that take such phenomena into account in a single joint model. Multi-seasonality can be modelled using TBATS models. The study confirmed that TBATS models can be considered useful alternatives in planning cash replenishments in ATM networks.

Contribution & Value Added: This article is an extensive empirical study on the selection of proper methods and forecasting models necessary to predict withdrawals from ATMs with overlapping multi-seasonalities and calendar effects. We proved that taking seasonal and calendar effects into account when forecasting withdrawals from ATMs significantly reduces forecast errors. Statistically significant improvement in forecast accuracy was observed both for SARIMA and TBATS. After taking calendar effects into account, TBATS forecast errors were slightly smaller than those resulting from corresponding SARIMA models. However, this result is statistically insignificant. The results of this study imply a need for further studies on the applications of TBATS models in forecasting the required cash level in ATMs, which in turn may help improve the efficiency of ATMs network management.

Article type:	research article
Keywords:	ATM; withdrawals; forecasts of withdrawals; ATM network management; TBATS model; SARIMA model; calendar effects
JEL codes:	C49, C53

Received: 24 October 2022 Revised: 26 April 2023 Accepted: 4 May 2023

Suggested citation:

Gurgul, H., Lach, Ł., Suder, M., & Szpyt, K. (2023). Using trigonometric seasonal models in forecasting the size of withdrawals from automated teller machines. *Entrepreneurial Business and Economics Review,* 11(3), 181-204. https://doi.org/10.15678/EBER.2023.110312

INTRODUCTION

Automated teller machines (ATMs) are computerized telecommunication devices. They provide the customers of financial institutions with a method of performing financial transactions in a public space without the need for a human clerk. Automated teller machines are part of a so-called 'cash supply chain,' 'cash chain,' or 'currency chain,' which generally consists of a Central Bank, mints/banknote printers, a distribution network, commercial banks, public customers, and businesses. A cash chain is characterized by the forward and backward motion of coins and notes to distribute cash to the public and ensure that the cash in circulation remains 'fit' (*i.e*., valid for circulation). The return flow of cash serves the purpose of removing all unfit cash (especially notes) and returning it to the Central Bank. The cash chain can be classified as a closed-loop supply chain because, ideally, over time, no cash leaves circulation. Automated teller machines are supplied/replenished by the distribution network, while the public or business customers demand/withdraw cash. Of the wide variety of ATM types currently deployed globally, the classic ATM with the ability to dispense only banknotes is still the most widespread.

The most recent reports of the National Bank of Poland (NBP; see the report of NBP for the first quarter of 2022) stress that withdrawals are becoming fewer in number. However, the value of individual transactions is becoming larger. This tendency, which has been observed in recent years, is not profitable for the operators of ATM networks. This situation has two negative consequences. First of all, the growing value of individual withdrawals means that operators must use more cash to replenish ATMs. Therefore, the costs of frozen cash and transporting it are higher. Secondly, the reduced number of withdrawals is a source of lower income from *interchange* and from the advertisements that are displayed on the screen for customers while they are withdrawing cash from ATMs. If this tendency continues then the revenues of ATM operators will fall. In this situation, the operators of ATM networks must try to reduce the costs of servicing the networks. A substantial proportion of these costs results from transporting cash and freezing it.

The costs of cash management can be as high as 50% of the total service costs of ATMs (Simutis *et al.*, 2007; Toi, 2011; Suder, 2015). The greatest challenge faced by ATM operators is to minimize the charges resulting from their cash service. At the same time, the operator tries to ensure that there is always an appropriate amount of cash in ATMs. In other words, the goal is to keep at an acceptably low level the probability that the user of an ATM faces a lack of cash in the ATM.

Cash management goals can be achieved if cash supplies in ATMs are properly managed. The replenishment process depends on proper, accurate forecasts of cash withdrawals from ATMs. Therefore, ATM operators, supported by researchers, are trying to find better forecast methods that will ensure better forecast quality. Thus, one research direction consists in testing the new models with respect to their effectiveness in forecasting cash demand. Some classes of models that are well-known in other areas of applied econometrics have not been verified or even used in cash withdrawals from ATMs. In particular, many special calendar effects have not been included in the forecasting models used.

The most frequently used techniques in forecasting withdrawals from ATMs are based on SARIMA models. These models take seasonality (*e.g.,* weekly seasonality) into account but do not reflect multi-seasonality in the same time interval (*e.g.,* daily, weekly, monthly, all in the same time interval). At the same time, cash withdrawals from ATMs display multi-seasonality. The overlapping seasonalities imply that when modelling withdrawals techniques that consider multi-seasonality in one model should be used. The multi-seasonality of processes is modelled in different fields of study by TBATS models but not in the ATM framework. Therefore, our task was to compare the forecasts of withdrawals by traditional SARIMA models to TBATS models (not used in modelling withdrawals from ATMs). Moreover, we verified the extent to which including seasonal and calendar effects improves forecast quality.

Our research was based on daily withdrawals from 74 ATMs managed by one of the largest ATMs networks in Poland. The data used encompassed the period between January 2017 and December 2019.

This article is divided into five sections. The next section will present the literature overview concerning the forecast of withdrawals from ATMs and the application of TBATS models in the forecasts of selected time series. Based on the literature, in the final part of this section, we will state our research hypotheses. In section three, we will describe the data used in the article. In particular, we will define the seasonal and calendar effects included in the models and present the models used in this study. In section four, we will show the computations results. In the last part of the article, section five, we will present the conclusions and limitations of this research. Replication codes accompanying the article are available in the Appendix at the end of the article.

LITERATURE REVIEW AND HYPOTHESES DEVELOPMENT

Over the last two decades, ATMs have been one of the most important sources of cash. However, the literature concerning the functioning of ATMs, logistics, convoy of cash, and cash management in ATM networks is rather scarce. Currently, as of 2 October 2022, the Scopus database lists less than 1000 articles that mention all aspects of ATMs' operations. Most of these contributions (almost 80%) consider issues concerning ATMs construction, the service of ATMs, and their functioning as tele-informatic devices. Contributions on the time series of ATM withdrawals make up 12% of the total number of articles on ATMs. The low number of articles on withdrawals is caused to a large extent by a lack of respective data.

Articles concerning the management of ATM networks mostly present two basic questions. One concentrates on the econometric properties of the time series of withdrawals with a special focus on seasonal and calendar effects. The other issue most frequently considered is the selection of proper methods and forecasting models necessary to predict withdrawals from ATMs.

An interesting article on seasonal and calendar effects was written by Rodrigues and Esteves (2010), who considered their influence on withdrawals from ATMs in Portugal. They took the following calendar effects into account: the day of the week, the week of the month, the month of the year, and the effects of church and national holidays. Rodrigues and Esteves stress the significance of these effects with respect to daily withdrawals from ATMs. Their results show the impact of seasonal and calendar effects on the structure of the time series of withdrawals from ATMs. The authors used the quarterly national accounts' procedure of adjusting data for seasonality and working days effects. This procedure allowed them to apply the ATM series as an instrument for forecasting private consumption.

Findley and Monsell's results (2009) are mostly in line with those of Rodrigues and Esteves (2010). Findley and Monsell established the impact of not only seasonal effects (day of week, week of month, month of year) but also church and national holidays. They suggest that in modelling and forecasting the time series of withdrawals, it is necessary to consider the specific day and month. The authors used X-12-ARIMA (Census Bureau's seasonal adjustment program).

The time series of withdrawals are the subject of an article by Cabrero *et al*. (2009). Among other issues, they investigated the daily cash demand from ATMs. They suggest daily, monthly, and yearly patterns. Cabrero *et al*. (2009) indicated payment patterns and customers behaviour. They emphasize the effect of the trading day (an increase in banknotes directly before the weekend and a decrease after weekends). Similarly, the number of banknotes is lower in the first half of the month and higher at the end of the month. Moreover, the authors draw attention to the unavoidable impact of holidays on the demand for cash from ATMs. Cabrero *et al*. (2009) compared two competing approaches to model seasonality in daily time series: the ARIMA-based approach and the structural time series approach.

Simutis *et al.* (2008), Takala and Viren (2007), and Toit (2011) indicate that withdrawals from ATMs can be impacted by paydays in institutions and firms, holiday periods, and other factors that determine trends and weekly, monthly, and yearly cycles. These calendar effects are important within the framework of the logistics of ATM networks.

Calendar effects in the time series of withdrawals are analyzed in articles by Gurgul and Suder (2012, 2016), and Suder (2015). They investigate calendar and seasonal effects on the time series of the number and size of withdrawals from the ATMs of one of the largest network operators in Poland. The results of their analyses are in line with the previously mentioned analyses. The authors stress that calendar effects can be observed in the time series of ATM withdrawals (both the size of particular withdrawals and the number of withdrawals). However, the impact of particular effects depends on an ATM's location.

In the articles reviewed so far, the authors focus on detecting calendar and seasonal effects and conclude that they should be taken into account when predicting ATM withdrawals. However, there are also contributions which try to include calendar and seasonal effects in models that forecast cash replenishments.

Wagner (2010) detected seasonal and calendar effects not only in the time series of cash withdrawals from the analyzed ATMs but also included these effects in his prediction models. He used the SARIMA and VAR models. His goal was to predict the time series of withdrawals from ATMs which belong to one European bank. He concluded that modelling special days (*i.e.* days with calendar or seasonal effects in day or month) using dummies is better than forecasting without taking calendar or seasonal effects into account. Forecast errors in extended models are essentially lower.

Gurgul and Suder (2013) apply SARIMA models and switching models to the forecast of withdrawals from ATMs and Bielak *et al*. (2015) describe the results of forecasting withdrawals from ATMs in bank branches using the ARMAX model and neural networks.

According to Rafi *et al*. (2021), the changing demands of users and changing seasonal patterns are very challenging problems for an ATM network. Financial institutions must fill each ATM with the optimal amount of cash. In this study, the authors used a time series model similar to ARIMA. Furthermore, this study used ATM data from six different financial organizations in Pakistan. There were 2040 distinct ATMs and 18 months' worth of replenishment data from these ATMs. The mean absolute percentage error (MAPE) and symmetric mean absolute percentage error (SMAPE) were used to evaluate the models. The suggested model, based on ARIMA, turned out to be better than models based on neural networks (lower MAPE and SMAPE in the case of ARIMA).

The goal of Fallahtafti *et al*. (2022) was to forecast the ATM cash demand for the periods both before and during the Covid-19 pandemic. Their other aim was to compare several statistical (based on ARIMA) and machine learning models in terms of different algorithms and assumptions. To achieve this goal, the ATMs were first clustered based on accessibility and surrounding environmental factors. These factors significantly affect the cash withdrawal pattern. The authors found that during Covid-19 and in times of shocks in demand and huge volatility of withdrawals, the statistical models (ARIMA and SARIMA) can provide better forecasts. According to the authors, the probable reason for this is that such models perform well, especially for short-term predictions. This kind of model allows overfitting to be minimized.

A relatively large number of authors who write about forecasting ATM withdrawals use neural networks. Including calendar effects, they are creating neural networks and show that their method is better than autoregression models. Interesting empirical studies on the applications of neural networks have been conducted by Simutis *et al.* (2007; 2008) and Acuña *et al.* (2012). They attempt to optimize cash management by selecting an appropriate withdrawal prediction model. The authors demonstrate that the use of neural networks can reduce network service costs by as much as 20%.

More recent articles on the use of neural networks in forecasting withdrawals from ATMs are by Serengil and Ozpinar (2019), Ekinci *et al.* (2015), Zandevakili and Javanmard (2014), Bhandari and Gill (2016). The promising computer-supported methods with respect to ATM applications may be based on evolutionary algorithms which are thoroughly presented in Sieja and Wach (2019). They also indicate the possibility of their implementation for the needs of the economy, especially the entrepreneurial economy.

To the best of our knowledge, previous studies have not suggested how to model usage or forecast withdrawals from ATMs using TBATS. Notably, we checked the Scopus and Google Scholar databases on 2 October 2022. This result is surprising, because this class of model is widely used in forecasting time series. Moreover, the authors of these contributions demonstrate that forecasts obtained from TBATS models are often more accurate than the forecasts derived from SARIMA models or neural networks. We used the *nnetar* function available in R package *forecast*. Please note that numerical experiments with neural networks convinced us that for our data set, SARIMA models produce more accurate predictions than neural networks. As a consequence, we focused on the comparison between TBATS and SARIMA.

Important contributions to TBATS theory and applications, as well as packages (in the R environment), have been developed by De Livera *et al.* (2011). Moreover, TBATS models are used as forecasting tools in various types of time series.

Perone (2022) compared ARIMA, exponential smoothing (ETS), neural network autoregression (NNAR), TBATS and hybrid models to forecast the second wave of Covid-19 hospitalizations in Italy. He found that the best single models were NNAR and ARIMA for patients from both groups of patients, depending on the severity of the illness. However, for patients with mild symptoms, the most accurate were the hybrid models NNAR–TBATS, ARIMA–NNAR, and ARIMA–NNAR–TBATS. Perone concludes that compared to the single models the hybrid statistical models capture a greater number of properties in the medical time series data structure. The predictions seem to suggest interesting practical implications.

Talkhi *et al.* (2021) consider similar forecasting questions with respect to Covid-19 in Iran and also use the TBATS model. This is not the only attempt to forecast using TBATS models. Applying TBATS, ARIMA, their statistical hybrid, and a mechanistic mathematical model combining the best of the previous models, Sardar *et al.* (2020) attempted to predict the daily confirmed cases of Covid-19 across India and in five different Indian states (Delhi, Gujarat, Maharashtra, Punjab, and Tamil Nadu) for the second half of May 2020. The ensemble model demonstrated the best prediction capacity and suggested that daily Covid-19 cases would significantly increase in the forecast window considered. Furthermore, the lockdown measures would be more effective in states with the highest percentages of symptomatic infection.

An interesting contribution using combined TBATS with the support vector machine (SVM) model of minimum and maximum air temperatures applied to wheat yield prediction at different locations in Europe is that of Gos *et al.* (2020). They found that the precision of air temperature prediction improves when using combined SVM/TBATS modelling, compared to pure TBATS or SVM modelling. Depending on the locations, which can be related to different climatic conditions, this improvement was between 3% and 14% for the maximum daily air temperature. The interval of improvement varied between 5% and 25% for the minimum daily air temperature.

Munim (2022) found that in modelling the container freight rate, the TBATS model or a combination of TBATS and SARIMA forecasts are better than the SARIMA and seasonal neural network autoregression (SNNAR) models as well as their combinations, both in training and test-sample forecasts. Munim emphasizes that none of the forecasting methods performs better than the TBATS model. Furthermore, for the robustness of the cross-validation, each test-sample data point is forecast using model re-estimation, which improves the forecast performance of the SARIMA and SNNAR models but not of TBATS.

It follows from this literature overview that there are seasonal and calendar effects in the time series of withdrawals from ATMs. They should be included in the forecast models. Moreover, the literature overview convinces us that TBATS models can be useful. Therefore, we formulated the following hypotheses:

H1: Taking seasonal and calendar effects into account when forecasting withdrawals from ATMs reduces forecast errors. The improvement of forecast accuracy is statistically significant and occurs for both SARIMA and TBATS.

H2: Forecast errors obtained in the TBATS model are smaller than those resulting from using the SARIMA model.

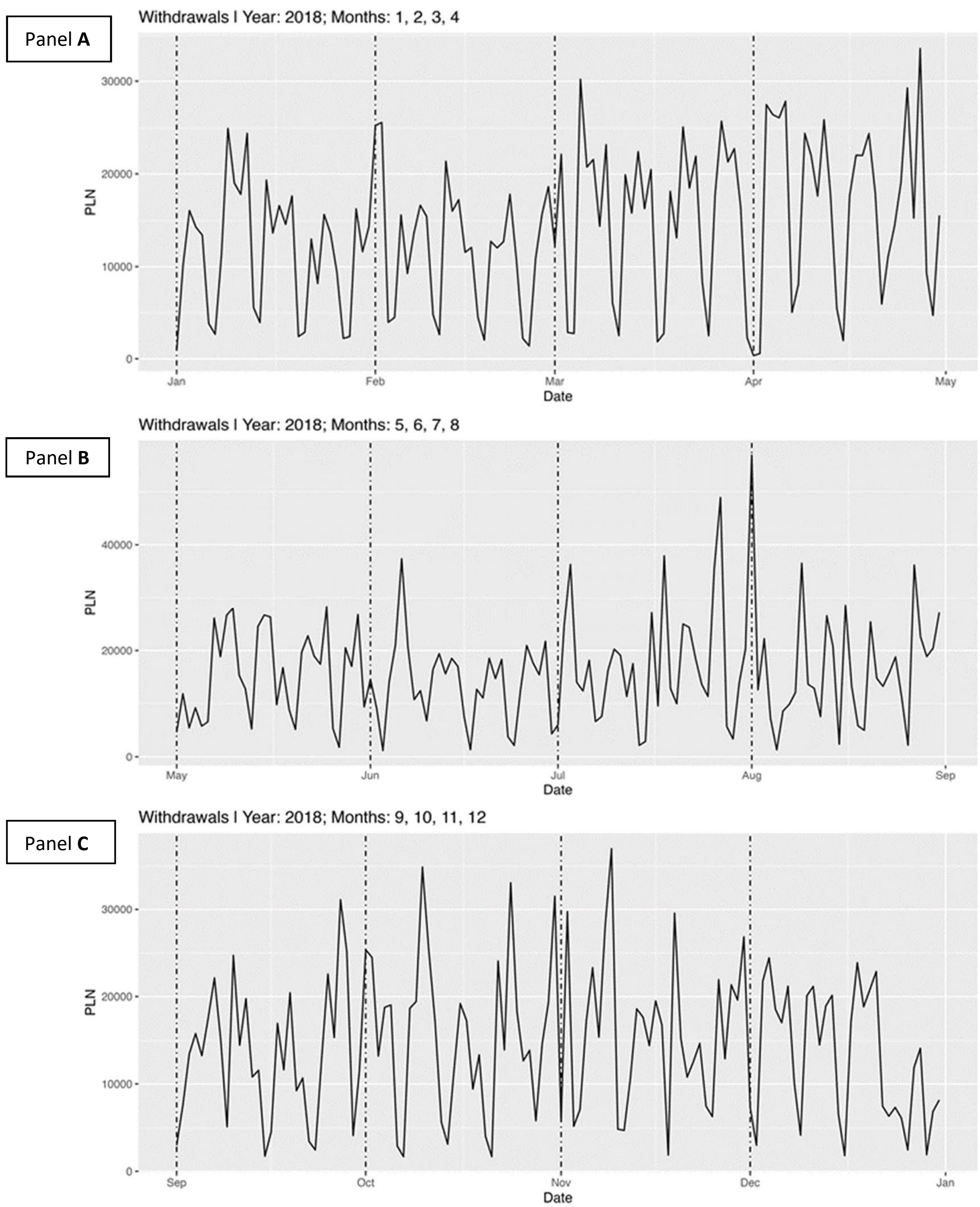
RESEARCH METHODOLOGY

In this part of the article, we will describe the structure of the data used in the analysis. We will also present the methodology and models used in the empirical part of the article. Moreover, we will define the procedure used to compare the models.

Sample and Data Collection

We analyzed the dataset on withdrawals from 74 ATMs covering the period from 1 January 2017 to 31 December 2019, *i.e.* before the outbreak of the Covid-19 pandemic. We wanted to verify the usefulness of TBATS models in ATM's withdrawal forecasting without facing the possible bias in the data caused by the pandemic.

Referring to other contributions, we checked the basic econometric properties of the time series of the withdrawals. To illustrate the dynamics of examined data, Figure 1 presents the main features of the time series of withdrawals from examined ATMs.[1]



[1] Access to the complete dataset used in this article is possible after contacting the corresponding author and obtaining acceptance of such a request by the data provider.

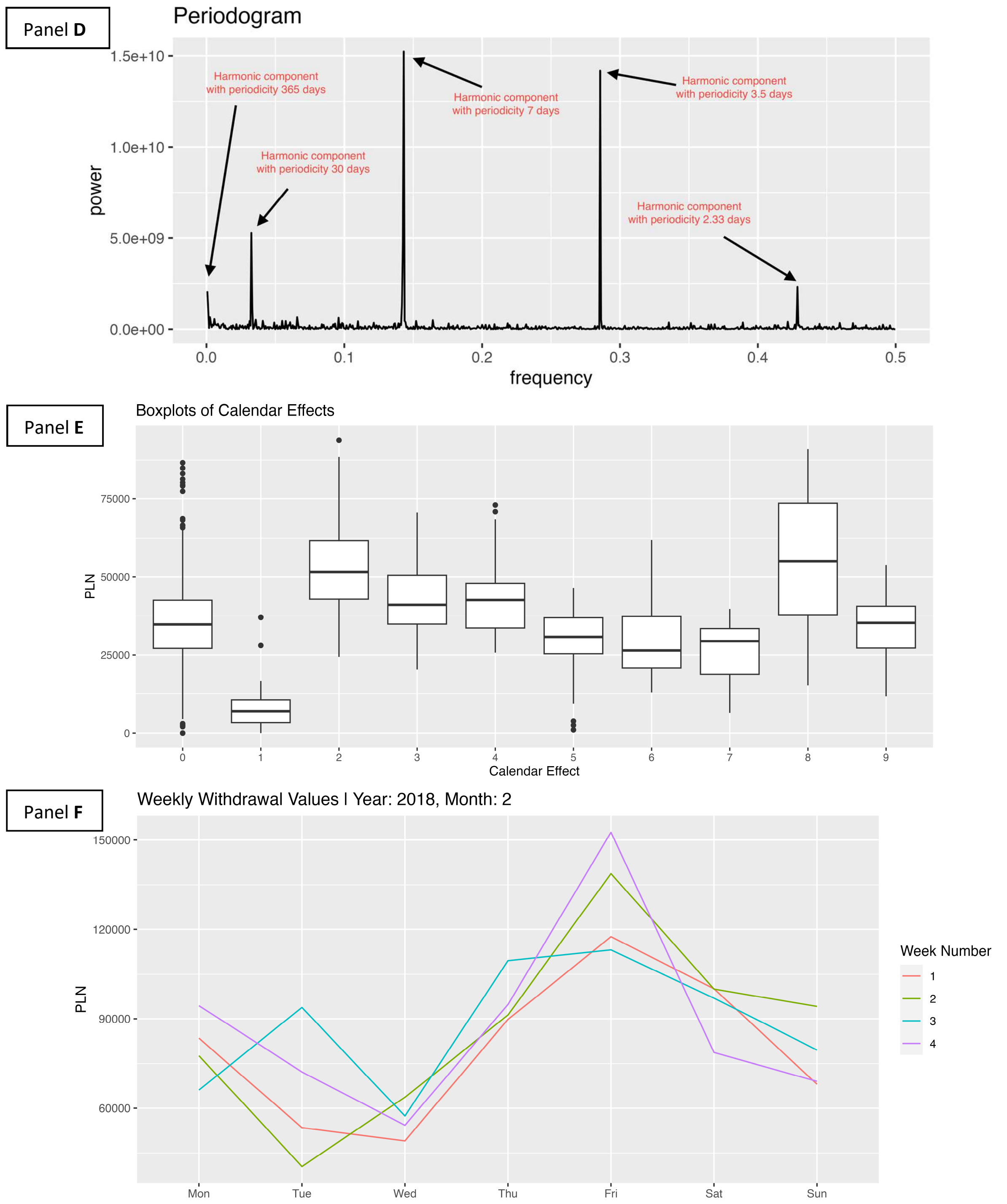


Figure 1. Main features of time series of ATM withdrawals

Note: Panels A-C show 2018 withdrawal data in one-month slices for a particular ATM, Panel D shows a raw periodogram for a particular ATM in the period 2017-2019, Panel E illustrates boxplots of withdrawals for selected nine types of special days (numbers 1-9 on the x-axis) and typical days (0 on the x-axis) for a particular ATM over the examined period, Panel F shows weekly withdrawal data in February 2018 for a particular ATM.

Source: own elaboration.

Figure 1 provides insights on the list of possible effects occurring in the ATM-related dataset. Panels A-C in Figure 1 present the exemplary ATM withdrawal data.[2] The presented time series clearly exhibits seasonal patterns. In the next step, we conducted trend analysis. The main reason for the presence of a deterministic trend in most of the analyzed series on ATM withdrawal was the general trend of decreasing demand for cash in 2017-2020 (see NPB report for the first quarter of 2022), which translated into a decrease in the total daily amount of withdrawals from ATMs. This type of trending behaviour was found for most of the examined ATMs. However, for some ATMs, there was a noticeable increase in the size of withdrawals. This may be due to the development of infrastructure in the neighbourhood of the ATM location or the removal of nearby ATMs. Unfortunately, due to the lack of information on the location of ATMs, it is not possible to verify the reason for this increase in demand for cash at a given ATM. After conducting linear and logistic trend analysis for the 2017-2019 sample, we found that, in general, the logistic trend not only fits best the ATM withdrawals data but at the same time the shape of the fitted trend line, *i.e.* logistic S-curve, seems to correspond to typical stages of ATM development (installation, fast development, stabilization). We did not find a single set of parameters of the trend function that would fit well to majority of the ATM's. A visual inspection of raw periodograms (comp. Figure 1, Panel D) allowed us to obtain insights on possible seasonal and calendar effects occurring in the time series under study. Statistically significant levels of ACF (at 5% level) were found for weekly, monthly, and yearly seasonality for all but one ATM[3] with the dominant pattern found for weekly seasonality (comp. Figure 1, Panels D and F).[4] After scanning the data (comp. Figure 1, Panel E), we decided that the following nine calendar effects should be taken into account in this article:

1. Work-free holidays (such as Easter and Christmas);
2. The tenth day of a month, when wages and salaries are most commonly paid in Poland;
3. The first day of the month;
4. The last day of the month;
5. Trading Sundays;
6. The beginning of a long weekend;
7. The end of a long weekend;
8. The day before the beginning of a long weekend;
9. The day after the end of a long weekend

The individual properties (*i.e.* trends, seasonal and calendar effects) of time series of ATM cash withdrawals were used to construct respective forecasting models, as we focused on out-of-sample forecasts in the study. Each effect presented in Figure 1 was derived based on withdrawal data for the selected ATM that strongly exhibited the given effect. Individual ATMs may vary in terms of intensity of a given effect, *e.g.,* in case of some ATMS the weekly seasonality may be more pronounced then in case of other ATMs. For example, Figure A2 in the appendix presents all the effects discussed in Figure 1 but this time derived for a single randomly selected ATM. The forecasting procedure was fully automated and implemented in the R environment.

Forecasting Models

In the econometric literature, backshift (B) notation is widely used when defining time series models. In particular, this operator allows one to write down ARIMA models more clearly and comprehensively. The operator B backshifts the values of the time series:

[2] For some examined ATMs, the respective time series on cash withdrawals contained some zeros that corresponded to various potential causes (*e.g.,* ATMs located in shops being closed on non-trade days, random ATM failures etc.). At the same time, the use of a Box-Cox transformation is limited to positive time series. To handle data with zero values, the estimation procedure of TBATS models implemented in the *tbats* function in *forecast* package used inverse hyperbolic sine transformation (Johnson, 1949). Moreover, when calculating the forecast errors, we skipped the dates with known zero withdrawals as such cases would spuriously improve the forecast accuracy.

[3] Complete results of the initial stage of the analysis are available from authors upon request.

[4] On the periodogram, one can also see shocks for frequency 0.284 and 0.428, which are called harmonic shocks and support existence of 3.5 and 2.33-day cycles, respectively. The one-week cycle is a multiple of these two cycles (Bloomfield, 2000).

$$BX_t = X_{t-1}, B^2X_t = X_{t-2}, \dots, B^nX_t = X_{t-n} \tag{1}$$

in which $(1-B)$ denotes the difference of order 1, $(1-B)^2$ denotes the second difference, and $(1-B)^n$ stands for the n^{th} difference. Taking backshift notation into account, we can define the $ARIMA(p,d,q)$ model in the following way:

$$(1-\phi_1 B - \dots - \phi_p B^p)X_t' = c + (1+\theta_1 B + \dots + \theta_q B^q)\varepsilon_t \tag{2}$$

in which X_t' =$(1-B)^d X_t$ denotes the integrated time series, $\phi_1, \dots, \phi_p$ are parameters of $AR(p)$ and $\theta_1, \dots, \theta_q$ are parameters of $MA(q)$, d is the order of differences, and ε_t is white noise.

The SARIMA models are a further generalization of the ARIMA models. The ARIMA models are not able to model data with seasonality. This drawback can be omitted by generalizing the ARIMA into seasonal autoregressive integrated moving average (SARIMA). Using the backshift operator $SARIMA(p,d,q)(P,D,Q)_m$ may be defined as follows:

$$\begin{aligned}(1-\phi_1 B - \dots - \phi_p B^p)(1-\Phi_1 B^m - \dots - \Phi_P B^{Pm})(1-B)^d(1-B^{Dm})X_t = \\ = c + (1+\theta_1 B + \dots + \theta_q B^q)(1+\Theta_1 B^m + \ \dots \ + \Theta_Q B^{Qm})\varepsilon_t\end{aligned} \tag{3}$$

In (3), the triple (p,d,q) stands for the nonseasonal component of the SARIMA model and $(P,D,Q)_m$ denotes seasonal part of the model. In detail, P stands for seasonal autoregressive order, D denotes seasonal difference order, Q stands for seasonal moving average order, and m denotes the number of time steps for a single seasonal period.[5]

The alternative group of models used in the empirical part of this article comprises the BATS and TBATS models.[6] The name BATS/TBATS is an acronym consisting of four/five letters: T – trigonometric seasonality, B – the Box-Cox transformation, A – ARIMA errors, T – Trend, and S – seasonal components.

The BATS model is a modification of exponential smoothing (double-seasonal Holt-Winters forecasting method). We supplemented this well-known method with the Box-Cox variance stabilizing transformation, ARMA model for the error term and inclusion of T seasonal patterns. The ARMA model allows one to remove the problem of autocorrelation in residuals. In short, the BATS model is a combination of exponential smoothing, the Box-Cox transformation, and errors of the ARMA process with several (not only two) seasonal components.

The following equations (4)-(9) describe the components of the BATS model (De Livera *et al.*, 2011):

$$X_t^{\omega} = \begin{cases} \frac{X_t^{\omega}-1}{\omega} & for\ \omega \neq 0 \\ log(X_t) & for\ \omega = 0 \end{cases} \tag{4}$$

$$X_t^{\omega} = l_{t-1} + \varphi b_{t-1} + \sum_{i=1}^{T} s_{t-m_i}^i + d_t \tag{5}$$

$$l_t = l_{t-1} + \varphi b_{t-1} + \alpha d_t \tag{6}$$

$$b_t = (1-\varphi)b + \varphi b_{t-1} + \beta d_t \tag{7}$$

$$s_t^i = s_{t-m_i}^i + \gamma_i d_t \tag{8}$$

$$d_t = \sum_{i=1}^{p} \phi_i d_{t-1} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t \tag{9}$$

in which:

l_t - denotes local level in time t;
b_t - component reflecting the short-term trend in time t;
b - parameter reflecting the long-term trend;
s_t^i - i-th seasonal component in time t, where $i = 1, \dots, T$;
d_t - the ARMA(p,q) process;
ε_t - white noise;
α, β, γ_i - smoothing parameters, where $i = 1, \dots, T$;

[5] Comprehensive theoretical background for time series modelling (including ARIMA/SARIMA models) is given in detail in Hyndman *et al.* (2008) and Shumway and Stoffer (2010).
[6] A detailed theoretical introduction to BATS/TBATS models is given in a textbook by Hyndman and Athanasopoulos (2021).

φ - parameter-suppressing trend.[7]

Equation (4) defines the Box-Cox transformation, where ω is a parameter of the transformation and X_t is observation at time t. Equation (5) is a mathematical formula for the BATS model, consisting of 4 main components described in (9)-(12). Equation (6) is a formula for the local level of time series, (7) is a formula for the short-term trend at time t, (8) defines the i-th seasonal component at time t and (9) shows the formula for $ARMA(p,q)$. The domain of the BATS model is given by expression (10):

$$BATS(\omega, \varphi, p, q, \{m_1, \dots, m_T\}) \tag{10}$$

in which:

ω - the parameter of the Box-Cox transformation;
p, q - time lags of $\mathrm{ARMA}(p,q)$;
φ - suppressing parameter;
m_i - the length of the i-th seasonal window, $i = 1, \dots, T$.

The TBATS model is a modification of the BATS model. The TBATS model employs the trigonometric model of seasonality. In BATS, it is assumed that m_i ($i = 1, \dots, T$) is a natural number. In TBATS, this parameter can be set to any positive real number. This extension is possible due to the inclusion of a trigonometric model of seasonality:

$$\begin{gathered} s_t^i = \sum_{j=1}^{k_i} s_{j,t}^i \\ s_{j,t}^i = s_{j,t-1}^i cos\lambda_j^i + s_{j,t-1}^{*i} sin\lambda_j^i + \gamma_1^i d_t \\ s_{j,t}^{*i} = -s_{j,t-1}^i sin\lambda_j^i + s_{j,t-1}^{*i} cos\lambda_j^i + \gamma_2^i d_t \end{gathered} \tag{11}$$

in which:

$s_{j,t}^i$ - stochastic increase in level in the i-th seasonal component, which is necessary to describe the change in the i-th seasonal component in time by means of $s_{j,t}^{*i}$, where $i = 1, \dots, T$, and $j = 1, \dots, k_i$;
γ_1^i, γ_2^i - smoothing parameters;
k_i - the number of harmonic components (at frequencies $\lambda_j^i = \frac{2\pi j}{m_i}$ where $j = 1, \dots, k_i$,) necessary for the i-th component seasonal window.[8] The approach is equivalent to index seasonal approaches when $k_i = \frac{m_i}{2}$ for even values of m_i, and when $k_i = \frac{m_i - 1}{2}$ for odd values of m_i.

To summarize, TBATS models are often used for forecasting for several reasons. The Box-Cox transformation makes it possible to stabilize the variance, which is an important advantage in empirical analyses. The model allows one to simultaneously take into account many seasonal components with integer and non-integer lengths of seasonal windows. Function *tbats* in *forecast* package allows for estimating the lengths of seasonal windows as described in De Livera *et al*. (2011). Similarly to BATS, TBATS also takes the autocorrelation of residuals into account.

The abbreviated version of the model with a defined domain is given by:

$$TBATS(\omega, \varphi, p, q, \{\{m_1, k_1\}, \dots, \{m_T, k_T\}\}) \tag{12}$$

in which:

ω - parameter of the Box-Cox transformation;
p, q - time lags of $\mathrm{ARMA}(p,q)$;
φ - smoothing parameters;
$\{m_i, k_i\}$ - a pair of two parameters: m_i – the length of the i-th seasonal window, k_i – number of Fourier terms for the i-th seasonal effect, where $i = 1, \dots, T$.

[7] Although all TBATS models examined in this study were fit to detrended ATM withdrawal data, we did not impose any restrictions on the trend/level related parameters in (6) and (7). From various alternatives tested, we decided to use logistic function to remove trend from the raw data and next estimate unrestricted TBATS. As we checked such an approach resulted in better forecast accuracy compared to the case of using TBATS estimated on raw data.

[8] We followed De Livera *et al*. (2011) to specify TBATS model selection procedure, including the choice of the number of harmonics k_i in the trigonometric models. For technical details see Boshnakov and Halliday (2022).

In our computations, we used the open-source statistical software R and IDE RStudio. To estimate SARIMA models, we used the R package *stats*. Calendar and special effects were modelled using dummies in the respective models (Harvey, 1989). The *sarima* function in the core *stats* package allows one to use dummy variables reflecting calendar effects and special days effects. Technically, the estimation of such models is based on using the so-called SARIMAX formulation (also referred to as mean-corrected formulation) of model (3) in which X_t is replaced with $Y_t = X_t - f(t)$, where $f(t)$ can depend on exogenous (e.g., dummy) variables.

To estimate TBATS models, we used the MLE-based estimation procedure described in detail in De Livera *et al*. (2011) and implemented in R in *tbats* function by Slava Razbash and Rob J. Hyndman. The function is available in the *forecast* package. One of the major limitations of all existing implementations of TBATS models, including the one available in *tbats* function in *forecast* package in R, is the lack of possibility of including exogenous variables. To some extent, this drawback may explain why TBATS have not been widely used in forecasting withdrawals from ATMs. To overcome this limitation, we followed a hybrid two-step approach: the TBATS models were estimated on the basis of time series on ATMs withdrawals with calendar and special days excluded. In the second step, the withdrawals on the calendar and special days were separately estimated using basic ARIMA-class models. In further parts of this article, forecasts obtained from such a two-step procedure will be referred to as forecasts from TBATS models extended with calendar and special days. The complete replication code in R illustrating full details on all the econometric models used in our study is available in the Appendix at the end of this article.

Methods of Forecast Comparison

There are standard measures which allow comparing accuracy of the forecasts of the time series obtained in different models. In the case of ATM data, the most popular absolute ex-post verification measures, such as mean squared error (MSE) or mean absolute error (MAE), are often not appropriate since ATM with extremely high levels of cash withdrawals will most likely also exhibit higher levels of absolute forecast errors compared to ATMs with relatively low average withdrawals. The reason is differences in the scale of the ATM-related time series being modelled. To support this claim, one could recall the case of two particular ATMs in the dataset – the one with the highest average daily cash withdrawal (denoted in H) and the one with lowest daily withdrawal (L). As we tested MAE for H were on average 2276% higher than their counterparts calculated for L. Figure A1 in the Appendix illustrates this phenomenon in detail by showing mean levels of monthly cash withdrawals with respective levels of MAE. To avoid or reduce the problem of scale in the case of ATMs with significantly different values, we used the following relative measures (Makridakis *et al*., 1998):

– mean absolute percentage error – MAPE:

$$MAPE(X_t, t_0, h) = \sum_{i=0}^{h-1} \left| \frac{X_{t_0+i} - \hat{X}_{t_0+i}}{X_{t_0+i}} \right| \tag{13}$$

– symmetric mean absolute percentage error – SMAPE:

$$SMAPE(X_t, t_0, h) = \sum_{i=0}^{h-1} \left| \frac{X_{t_0+i} - \hat{X}_{t_0+i}}{\frac{1}{2}(X_{t_0+i} + \hat{X}_{t_0+i})} \right| \tag{14}$$

in which:

X_{t_0+i} - actual value at time $t_0 + i$;

$\hat{X}_{t_0+i}$ - forecasted value at time $t_0 + i$;

h - forecast horizon;

t_0 - start of the forecast window.

Although the MAPE measure is one of the most popular ones, it also has some drawbacks. The most serious one is its asymmetry. Mean absolute percentage error gives higher results in the case of overestimated forecasts which may happen if, for example, a given ATM unexpectedly stops working for some time. In such a situation, the denominator in (13) becomes very low and MAPE suggests that the forecasting properties of the underlying model are poor, although this effect is due to an unexpected event, not the model itself. This drawback was removed in a modification of MAPE called symmetric MAPE (SMAPE). Symmetric MAPE also has some drawbacks. It is a proper measure only when both actual values and forecasted values exhibit the same sign. An additional reason for choosing SMAPE is the fact that – contrary to MAPE – this measure is used by the network operators which provided the data for this research.

The accuracy of the forecasts obtained using the SARIMA/TBATS models (both with and without calendar effects) was tested by comparing forecast errors with a one-month and two-week (14 days) forecast horizon. One-month forecasting periods are used by the data provider in the process of managing the ATM network. Therefore, this period was chosen in order to compare the results of our study with internal forecasts of the data providing company. In addition, 14-day forecasts were given attention in our study because in case of ATMs, which are replenished more than once a week, such a forecast horizon seems useful from the operational point of view. To take yearly seasonality into account, the estimation window was set as a two-year period preceding the forecast window, *i.e.* the estimation window for each model included 730 observations. Monthly forecasts were conducted for full months (*i.e.*, for each forecasted month the parameter h was set equal to the number of days in a given month and t_0 was set to the first day of the given month in (13) and (14)). Two-week forecasts were conducted for 12 selected 14-day periods. The periods were chosen in a way that they included different types of calendar effects and other special effects (*i.e.*, for each forecast the h was set equal to 14 and t_0 was set equal to a given starting day of the forecast windows listed in Table 2). To describe periods of the forecasts, we included information on the types of special days in Table 1 and Table 2. Given the specification shown in Table 1 and Table 2 one may claim that analysis of results of testing the accuracy of forecasts can help find the models that give best forecasts over periods with different types of calendar effects.

Table 1. Characteristics of monthly forecasts

Period number	Forecasts periods	Calendar effects in the given periods
1	January	One one-day church holiday, winter holiday, one Sunday trading day
2	February	One Sunday trading day
3	March	One Sunday trading day
4	April	Easter holiday, two Sunday trading days
5	May	Long weekend, one Sunday trading day
6	June	Long weekend, beginning of summer holidays, one Sunday trading day
7	July	Summer holiday, one Sunday trading day
8	August	Summer holiday, the beginning of the month, long weekend
9	September	Beginning of the school year
10	October	Beginning of the academic year
11	November	Two long holidays because of church and national holidays
12	December	Christmas holidays, trading Sundays

Note: Only special events in a given month are listed in the above table. We omitted events occurring every month *i.e.* one trading Sunday, the first and the last day of the month, and the tenth day of the month.
Source: own elaboration.

To verify our hypotheses concerning the accuracy of the forecasts obtained by means of TBATS and SARIMA with calendar effects, we compared the descriptive statistics for MAPE and SMAPE. These comparisons were made separately for forecasts in the selected periods and also jointly for all forecasts. The analysis of four models (*i.e.* the basic SARIMA model (denoted by S), the SARIMA model taking calendar and special effects into account (SARIMA with dummies; the model is denoted by S_CE), the basic TBATS (T) model and basic TBATS model extended with calendar and special effects (T_CE)), involved a comparison of the ex-post accuracy of forecast error. Moreover, the forecast errors for each ATM during the whole period under consideration were compared. The sample included 74x12=888 MAPE errors and the same number of SMAPE errors for monthly forecasts. Analogous analysis was performed for two-week forecasts. The comparisons were made in pairs of models, *i.e.* S_CE vs S, T vs S, T_CE vs T and T_CE vs S_CE. Focusing on these pairs enables formulating conclusions about the feasibility of the methods used in forecasting ATM withdrawal data.

Table 2. Characteristics of two-week forecasts

Period number	Forecasts periods	Calendar effects in the given periods
1	17.01-30.01	End of the month, trading Sunday, occasional holidays
2	5.02-18.02	10th day of the month, occasional holiday
3	25.03-7.04	Turn of the month, trading Sunday
4	11.04-24.04	Easter holidays
5	1.05-14.05	Beginning of the month, 10th day of the month, long weekend
6	17.06-30.06	Long weekend, Beginning of the summer holidays, trading Sunday
7	6.07-19.07	Summer holidays
8	1.08-14.08	Summer holidays, beginning of the month, every 10th day of the month
9	26.08-8.09	Turn of the month, beginning of the school year
10	01.10-14.10	Beginning of the month, beginning of the academic year
11	28.10-11.11	Turn of the month, two long weekends
12	16.12-29.12	Christmas period, trading Sundays

Source: own elaboration.

From the point of view of ATMs management, the best model should ensure the highest accuracy of out-of-sample forecasts as this may help to define strategies allowing for minimalization of operational costs. On the other hand, model specification is typically based on analyzing in-sample fit. In this study, we examined both of these issues. However, we paid attention mainly to checking the accuracy of out-of-sample forecasts, because this criterion is used in practice by managers of ATM networks.

RESULTS AND DISCUSSION

The results of the empirical analysis will be presented with respect to the accuracy measures and horizons of the forecasts. First of all, the statistics for forecast errors will be presented and evaluated. Then, the results of the comparison of forecast accuracy for each ATM will be summarized, depending on the type of error and the forecast horizon.

Comparison of MAPE for Monthly Forecasting

Table 3 presents the basic descriptive statistics of MAPE for monthly forecasts for 74 ATMs. To make the data easier to read, Figure 2 additionally shows a radar chart of mean MAPE calculated in selected months with respect to the type of model applied.

Table 3. Descriptive statistics of MAPE for monthly forecasts

Months	Mean				Median				Standard deviation			
	S [%]	S_CE [%]	T [%]	T_CE [%]	S [%]	S_CE [%]	T [%]	T_CE [%]	S [%]	S_CE [%]	T [%]	T_CE [%]
January	48.53	39.14	39.41	35.22	54.23	38.00	38.67	33.55	23.3	25.3	24.5	21.57
February	34.87	30.88	30.01	29.86	32.15	26.89	27.51	27.37	18.9	19.8	18.5	15.79
March	33.31	29.84	31.22	31.36	32.41	27.20	29.91	29.34	20.7	20.3	21.1	19.41
April	51.79	35.88	42.80	33.22	61.70	33.53	44.82	30.90	26.1	20.1	25.6	19.95
May	44.68	32.67	42.27	32.34	47.42	32.14	41.30	30.81	25	20.1	19.5	16.81
June	40.83	32.16	33.85	29.93	39.38	29.55	34.29	29.21	18.9	13.6	20.9	17.61
July	36.92	31.17	31.04	30.61	37.21	30.32	28.89	32.52	22.1	21.4	20.5	21.41
August	39.50	32.52	34.21	29.45	38.03	29.12	31.45	27.33	21.7	19	20.7	17.82
September	33.49	29.38	30.26	29.14	33.14	25.95	29.31	26.21	20.5	16.8	12.8	11.74
October	36.29	29.21	31.73	29.99	35.97	27.85	27.87	26.64	22.6	22	17.5	15.07
November	47.97	33.53	41.93	31.55	47.04	30.95	42.25	30.91	19.9	16.9	20.3	16.52
December	53.73	43.76	52.11	40.64	85.07	46.21	66.71	41.77	25.5	24.7	17.5	13.51
Total	41.82	33.34	36.74	31.94	45.31	31.48	36.91	30.55	22.10	20.01	19.95	17.27

Note: The lowest values for each month and each descriptive statistics are highlighted.
Source: own elaboration.

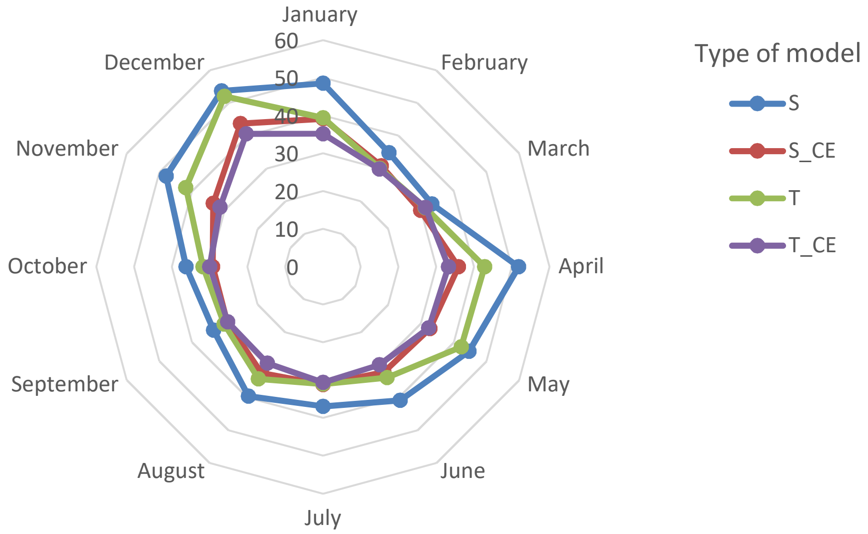


Figure 2. Radar chart for mean MAPE values (in %) in each month for different methods
Source: own elaboration.

Analysis of Table 3 and Figure 2 allows one to conclude that the lowest values of relative forecasts error were usually found for TBATS model extended with calendar and special effects. The exception was the forecasts for March and October. The forecasting performance of other models, *i.e.* those not taking calendar effects into account, was visibly worse as the lowest relative forecast error was never achieved for such models. The average MAPE calculated for all forecasts (*i.e.* for all months) was lowest in the case of the T_SE model and did not exceed 32%. It was about 1.4% lower than average MAPE calculated for S_CE model and almost 5% lower than the mean value of MAPE for the basic TBATS model. From a comparison of median of MAPE for 74 ATMs, it follows that the minimal median was also usually obtained for the T_CE model. For this model, the lowest median of MAPE was achieved in eight out of 12 months. In the case of three out of the remaining four months, the best model (in terms of the lowest median value of MAPE) was S_CE, and in only one month (July) the basic TBATS model was found to have the lowest median of MAPE. We should emphasize that the T_CE was found to provide more accurate forecasts in months with a smaller number of special days (*i.e.* March or October). The standard deviation of MAPE errors suggests that this measure exhibits the lowest volatility for monthly forecasts obtained via T_CE. The volatility of MAPE for the basic TBATS model is lower than for SARIMA. Considering calendar and special days effects visibly reduces the average MAPE value both in SARIMA and TBATS models. In the case of the SARIMA model, the inclusion of calendar effects and special days effects decreased the mean value of relative forecast errors by approximately 8% and in the case of TBATS – by around 5%.

To summarize, the basic TBATS model and TBATS model extended with calendar and special effects perform better with respect to forecasting daily withdrawals from ATMs than the SARIMA model (comp. Figure 2). We showed that including dummies (*i.e.* taking calendar and special effects into account) considerably reduces average errors in both SARIMA and TBATS frameworks. To test the statistical significance of these results, we conducted a series of Friedman tests applied to test the significance of differences in mean MAPE obtained from four pairs of models: T and S, T and T_CE, S and S_CE and finally T_CE and S_CE. The tests confirmed statistically significant differences between mean MAPE in the case of all pairs of models compared, except the pair T_CE and S_CE.[9] These results imply that basic TBATS allows obtaining more accurate forecasts compared to basic SARIMA. Moreover, they provide grounds for claiming that taking seasonal and calendar effects into account when forecasting withdrawals from ATMs significantly reduces forecast errors. Significant improvement in forecast quality was observed both for SARIMA and TBATS. The TBATS forecasts errors are slightly smaller than those resulting from SARIMA models. However, this result was found statistically insignificant (p-value in Friedman test equal to 0.42).

[9] Complete results of conducting Friedman test for all forecasts horizons are available from the authors upon request.

Comparison of SMAPE for Monthly Forecasts

To some extent, values of SMAPE for the monthly forecasts presented in Table 4 and Figure 3 confirmed the conclusions driven after the analysis of levels of MAPE. As far as SMAPE is concerned, the best monthly forecast (the lowest average relative forecast error) was obtained using TBATS model extended with calendar and special effects. The global average value of SMAPE for this model was 29.45% and was 1.5% lower than the corresponding average for forecasts obtained using S_CE, and over 2% lower than its counterpart obtained from the TBATS model. In this context, SARIMA's performance was the worst. Analyzing forecasts for particular months, we can formulate slightly different conclusions compared to the case of MAPE. The T_CE was the best with respect to SMAPE in the case of the nine months under consideration. However, in the other three months, basic TBATS was better. In none of the cases tested the minimal value of SMAPE was found for SARIMA models. In the case of the median of SMAPE, slightly better results were obtained using SARIMA with calendar effects.

Table 4. Descriptive statistics of SMAPE for monthly forecasts

Months	Mean				Median				Standard deviation			
	S [%]	S_CE [%]	T [%]	T_CE [%]	S [%]	S_CE [%]	T [%]	T_CE [%]	S [%]	S_CE [%]	T [%]	T_CE [%]
January	45.86	33.36	31.92	31.69	46.23	32.10	30.42	29.86	18.08	19.64	13.65	11.13
February	35.50	30.63	28.65	29.78	46.23	32.10	30.42	29.86	17.87	20.61	17.91	10.69
March	34.23	30.86	28.75	29.07	32.70	29.50	26.60	27.88	17.82	18.50	21.27	12.17
April	43.48	33.69	34.20	30.56	42.46	31.01	33.73	28.81	11.87	10.75	16.31	10.73
May	42.59	33.85	34.50	31.95	40.75	32.92	32.91	29.88	13.52	13.62	16.24	11.15
June	38.76	32.44	31.85	29.68	38.16	32.46	29.67	28.98	11.90	11.63	14.57	8.03
July	35.05	28.71	27.74	28.41	32.91	27.67	27.42	27.37	13.17	12.11	17.15	10.04
August	34.78	29.29	30.96	26.91	33.70	27.86	27.90	25.79	13.48	12.65	13.09	9.75
September	33.15	27.47	29.81	26.93	33.16	25.52	27.49	25.74	10.27	9.86	15.11	8.60
October	31.61	26.83	29.08	26.50	30.85	25.26	28.13	25.39	11.88	11.94	13.31	7.61
November	37.41	29.44	31.97	27.40	35.02	27.77	30.65	25.62	9.16	10.90	12.94	8.81
December	43.01	35.03	39.75	34.52	42.11	32.98	38.65	33.98	11.27	11.08	13.43	8.92
Total	37.95	30.97	31.60	29.45	37.86	29.76	30.33	28.26	13.36	13.61	15.42	9.80

Note: In the Table the lowest values are highlighted.
Source: own elaboration.

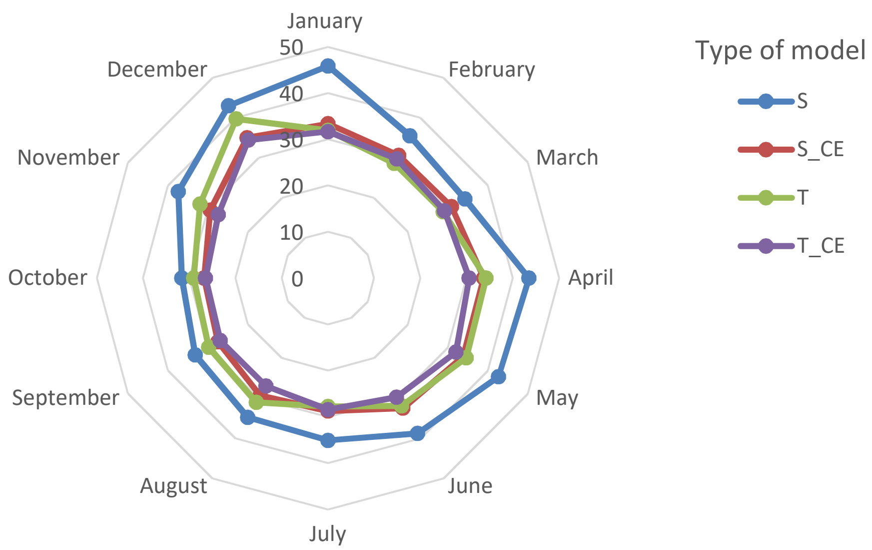


Figure 3. Radar chart for the means of SMAPE (in %) in particular months
Source: own elaboration.

In the case of September, October, and December, the median of SMAPE was the lowest for S_CE. However, these values were similar to the corresponding values obtained from T_CE. Global standard deviation of SMAPE was lowest in the case of the T_CE model and reached the level of 9.8% – which is considerably lower than in the other three models.

It follows from this part of the analysis that calendar and special effects have an important impact on the accuracy of forecasts. Therefore, one may claim that these effects should be taken into account in modelling ATM withdrawals.

We demonstrated that usually T_CE provides slightly better forecasts than S_CE. However, in February, March, and July (in these months there are only a few calendar effects) the basic TBATS model is more accurate than TBATS model extended with calendar and special effects. This convinces us that in the case of periods without strong calendar or special effects, basic models with no additional exogenous variables seem better alternatives for the purpose of forecasting. Similarly to the case of MAPE, we run a series of Friedman tests to check statistical significance of differences between mean SMAPE obtained from particular models. Only in the case of the pair S and S_CE the means of SMAPE were significantly different.

Comparison of MAPE for Two-week Forecasts

The results for two-week forecasts (Table 5, Figure 4) differed considerably from those obtained for forecasts with a monthly horizon. For two-week forecasts, SARIMA with dummies gave better results than T_CE. Both mean and median values of MAPE from this model showed the lowest value in seven periods under consideration. This effect was most visible at the turn of August and September and at the beginning of October. The difference in accuracy of forecast was approximately 5% in favor of S_CE model. The reason behind this phenomenon is probably the effects of the beginning of the school year and academic year – these are not seasonal. Global means and medians of MAPE are similar for S_CE and T_CE. Among models tested, the average standard deviation of MAPE of T_CE is the lowest. However, taking particular periods into account MAPE obtained in T_CE is smallest just in six out of 12 cases. Similarly to the case of monthly MAPE, we run a series of Friedman tests to check the statistical significance of differences between mean two-week MAPE obtained from particular models. The results led to similar conclusions as in the case of monthly MAPE.

Table 5. Descriptive statistics of MAPE for two-week forecasts

Period	Mean				Median				Standard deviation			
	S [%]	S_CE [%]	T [%]	T_CE [%]	S [%]	S_CE [%]	T [%]	T_CE [%]	S [%]	S_CE [%]	T [%]	T_CE [%]
17.01-30.01	47.53	35.34	35.03	33.79	50.18	35.28	35.88	30.13	24.56	23.37	22.66	21.31
5.02-18.02	42.98	45.16	32.83	26.69	41.53	46.98	30.37	24.75	19.86	21.12	17.89	17.52
25.03-7.04	32.14	28.63	34.18	31.93	30.28	27.45	32.11	29.64	16.87	17.77	17.14	19.25
11.04-24.04	47.12	39.98	50.04	40.90	107.41	40.30	73.22	41.35	121.43	74.99	27.86	22.70
1.05-14.05	52.77	35.42	52.09	37.91	62.14	33.34	59.66	36.59	24.66	19.12	24.81	19.62
17.06-30.06	41.45	31.35	44.15	32.56	43.84	29.02	45.19	29.70	26.24	15.88	21.78	14.67
6.07-19.07	34.46	28.75	37.73	38.01	34.35	29.14	36.58	37.78	22.09	21.49	24.24	23.20
1.08-14.08	41.08	26.86	33.44	25.66	41.05	25.10	32.88	25.06	24.25	22.16	21.49	22.43
26.08-8.09	32.18	27.48	34.36	32.64	31.73	26.40	34.40	32.77	20.61	21.60	20.96	20.83
01.10-14.10	33.29	27.15	36.59	32.53	30.22	24.25	32.48	28.98	18.13	15.95	18.59	16.92
28.10-11.11	53.90	33.02	47.48	32.54	66.51	31.33	54.62	31.46	25.04	19.81	26.41	20.97
16.12-29.12	52.02	51.95	53.29	51.18	149.31	60.96	122.46	60.62	19.84	25.60	22.76	25.62
Total	42.58	34.26	40.93	34.69	57.38	34.13	49.15	34.07	30.30	24.91	22.22	20.42

Note: In the Table, the smallest values for each forecasting period and each descriptive statistic are highlighted.
Source: own elaboration.

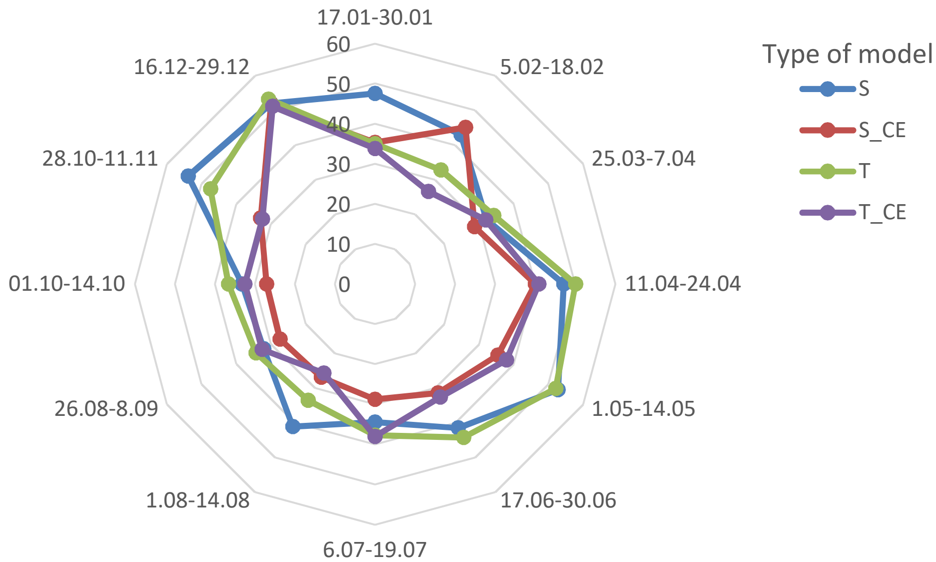


Figure 4. Radar chart of mean values of MAPE (in %) for two-week forecasts
Source: own elaboration.

Comparison of SMAPE for Two-week Forecasts

Global mean SMAPE for two-week forecasts was the smallest for SARIMA models with dummies (Table 6, Figure 5). For the S_CE model, the means of SMAPE for particular periods were only slightly (approximately 0.1%) lower than the corresponding values obtained from the T_CE model. However, the S_CE model turned out to be more accurate than T_CE in seven out of 12 cases. A similar regularity was found when comparing medians of SMAPE. It is in favour of the T_CE models that the volatility of errors was lower than its counterpart obtained from S_CE in nine out of 12 forecast periods.

Table 6. Descriptive statistics of SMAPE for two-week forecasts

Period	Mean				Median				Standard deviation			
	S [%]	S_CE [%]	T [%]	T_CE [%]	S [%]	S_CE [%]	T [%]	T_CE [%]	S [%]	S_CE [%]	T [%]	T_CE [%]
17.01-30.01	39.25	31.52	32.46	30.07	38.00	29.30	32.25	27.16	17.82	16.86	18.24	9.01
5.02-18.02	42.10	37.48	32.87	26.89	39.83	36.40	30.21	25.32	20.23	12.71	12.12	9.13
25.03-7.04	33.26	29.95	35.21	30.92	31.42	27.61	33.99	29.32	12.63	10.97	17.27	10.40
11.04-24.04	53.57	35.08	45.36	34.53	55.70	32.81	45.80	33.19	14.73	11.88	15.26	12.35
1.05-14.05	50.26	37.47	41.92	37.47	53.15	37.05	40.17	36.52	15.90	13.57	16.05	12.91
17.06-30.06	37.16	30.81	39.19	31.91	36.21	28.82	36.45	29.30	13.51	13.02	14.83	13.16
6.07-19.07	34.07	27.92	33.58	33.44	33.43	26.54	32.60	32.53	15.02	11.40	17.99	11.61
1.08-14.08	39.79	28.19	34.34	26.82	38.05	24.94	29.98	24.19	17.29	14.46	13.55	10.46
26.08-8.09	33.07	27.82	35.22	30.83	31.82	24.31	32.84	30.05	11.36	11.50	15.32	8.91
01.10-14.10	31.12	26.57	34.16	30.77	30.43	24.52	32.49	27.67	12.66	12.67	15.44	10.24
28.10-11.11	48.47	32.34	39.72	32.69	47.64	31.41	39.65	30.27	13.35	12.33	15.89	10.34
16.12-29.12	59.47	43.98	53.54	44.05	60.57	42.63	56.03	44.70	14.61	10.60	17.48	9.08
Total	41.80	32.43	38.13	32.53	41.36	30.53	36.87	30.85	14.92	12.66	15.79	10.63

Note: In the Table the lowest values are highlighted.
Source: own elaboration.

Based on the presented comparison of statistics of relative forecast errors, we may see that taking calendar effects (dummies) in the SARIMA model into account significantly improved the forecast accuracy for the horizons analyzed. Extending TBATS with calendar and special days effects improved the forecast accuracy in the case of two-week forecasts. Comparing SARIMA and TBATS both without dummies (not taking calendar effects into account) implies that TBATS-based forecasts are more accurate than the SARIMA-based ones. One of possible explanations is that TBATS allows modelling multiple seasonal patterns at the same time, it is not limited to just one pattern as in SARIMA. In the case of both models extended with dummies, the results were not unequivocal. In the case of monthly forecasts, we found smaller forecast errors for the TBATS model extended with calendar and special effects. However, in the case of two-week forecasts, SARIMA with dummies provided more accurate forecasts. Analogously to the case of two-week MAPE, we run the series of Friedman tests and obtained similar conclusions.

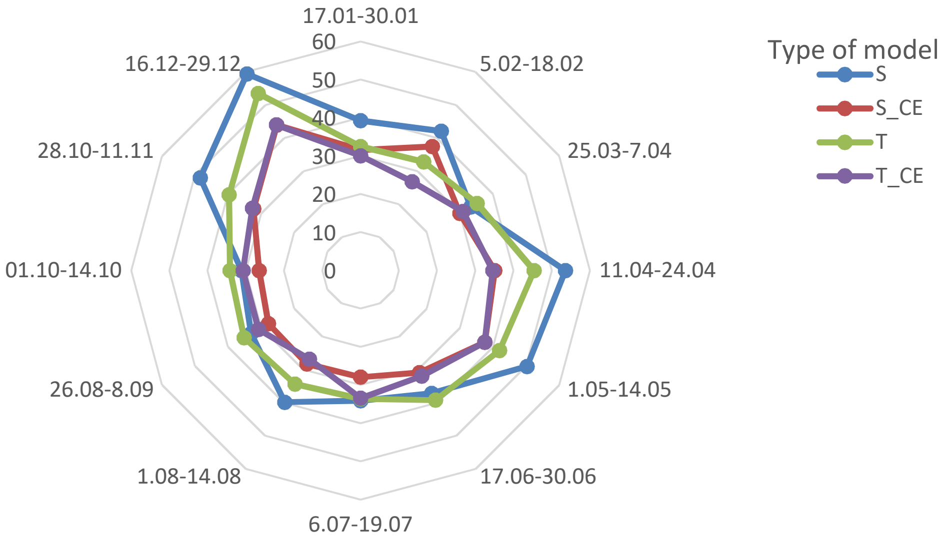


Figure 5. Radar chart of mean values of SMAPE (in %) for two-week forecasts

Source: own elaboration.

Results of Comparing Forecasts for Particular ATMs

The results in the previous sections concerning the quality of the models used to forecast withdrawals were not unequivocal. Therefore, we conducted additional, extended analyses to evaluate the feasibility of the methods and models. This analysis allowed us to compare the MAPE and SMAPE of forecasts for each ATM and each period. A comparison of forecasts was made for four pairs of models. The results are presented in Table 7.

Table 7. Comparison of the pairs of models

Comparison of the pairs of models	Monthly forecasts		Two weeks forecasts	
	MAPE	SMAPE	MAPE	SMAPE
T *vs S*	73.3%	76.9%	66.2%	68.5%
T_CE *vs* S_CE	59.3%	61.4%	48.4%	51.8%
S_CE *vs* S	95.7%	97.1%	92.9%	92.7%
T_CE *vs* T	75.8%	69.9%	83.0%	84.0%

Source: own elaboration.

As one can see in Table 7, monthly MAPE obtained from TBATS forecasts were better in 73.3 % pairs. This effect was even stronger for SMAPE (76.9%). In the case of two-week forecasts, TBATS also provided more accurate forecasts than SARIMA (the difference is about 7%-8%). Compared to S_CE the T_CE model turned out more accurate in about 60% of pairs (MAPE and SMAPE). A comparison of the accuracy of models taking calendar effects into account, with the basic models, *i.e.* pairs. S_CE *vs* S and T_CE *vs* T, shows that extended models are significantly better than basic ones. In more than 90% of cases tested, SARIMA models with dummies are more accurate than basic SARIMA. As we showed in previous sections, these regularities were confirmed by Friedman test.

To summarize, the accuracy of forecasts of SARIMA and TBATS extended with calendar effects was similar. Only in particular cases, the TBATS model extended with calendar and special effects outperformed SARIMA in terms of forecast accuracy, albeit this result was not found to be statistically significant.

CONCLUSIONS

Forecasting withdrawals from ATMs is an important part of managing an ATM network. Managers of ATM networks try to implement solutions that improve forecasting procedures to reduce the cost of replenishing ATMs. Proper forecasts of the cash necessary for ATMs reduce the operational costs of the networks.

In this study, a possible use of TBATS models to forecast withdrawals from ATMs was investigated. In practice, the SARIMA model is widely used as a forecasting tool. Our goal was to compare the SARIMA model with the TBATS model, both estimated with and without taking into account calendar and special effects. The main focus of this study was to verify the impact of calendar and seasonal effects on the forecast accuracy. We identified calendar and seasonal events (*e.g.,* holidays) and attempted to establish their impact on the forecast accuracy.

The empirical results confirmed that in most cases extended models are more accurate (lower MAPE and SMAPE values) compared to basic models. The hypothesis that the TBATS model is better (*i.e.* ensures lower values of MAPE and SMAPE) was verified only to some extent. When calendar effects were omitted, TBATS delivered forecasts of considerably better accuracy than SARIMA. After taking calendar effects into account the TBATS model was only slightly better than SARIMA (lack of statistical significance). Noteworthy, the TBATS model is used rarely as a forecasting tool by ATM networks. The results of this study may encourage further studies on the applications of TBATS models for forecasting the cash necessary in ATMs and in this way improve the network management. One observation is worth to be noted here. As we proved the forecasts obtained from basic TBATS models are in general significantly more accurate compared to the forecasts from basic SARIMA models. At the same time, the accuracy of the forecasts was nearly identical when calendar and special days are taken into consideration. Since existing implementations of TBATS models in statistical software do not allow for inclusion of exogenous variables, one could follow the idea of the hybrid approach presented in this article to incorporate the calendar and special days effects into TBATS framework. In our opinion, a desired direction of further research would be to focus on extending the range of possible formulations of TBATS models available in widely-used econometric software (*e.g*., R), especially in terms of the inclusion of time dummies.

This research suffers from several other limitations. The accuracy of forecasts may have depended on the location of the ATM. We can hypothesize that in some types of locations taking calendar days (*e.g.,* the day of the month) into account can reduce forecast errors. Because of the lack of data on the location of individual ATMs, we were unable to check the impact of this parameter on the improvement of forecast accuracy.

Another limitation of this research is the fact that the data came from only one city. Future research should be conducted to verify the possible applications of the models in particular types of locations. Finally, an interesting direction of future research would be to study the benefits of analyzing the usefulness of probabilistic forecasts (not only point forecasts as in this article) in the problem of managing ATMs network.

This study may inspire further, more complex analysis of TBATS models with respect to employing this type of model in forecasting cash demand in ATMs. This type of research may be useful for managers who estimate the demand for cash in ATM networks and branches of banks.

REFERENCES

Acuña, G., Cristián, R., & Millaray, C. (2012). Comparing NARX and NARMAX models using ANN and SM for cash demand forecasting for ATM In *The 2012 International Joint Conference Neural Networks (IJCNN)*, https://doi.org/10.1109/IJCNN.2012.6252476

Bhandari, R., & Gill, J. (2016). An artificial intelligence ATM forecasting system for hybrid neural networks. *International Journal of Computer Applications, 133*(3), 13-16. https://doi.org/10.5120/ijca2016907770

Bielak, J., Burda, A., Kowerski, M., & Pancerz, K. (2015). Modelling and Forecasting Cash Withdrawals in the Bank. *Barometr Regionalny*, *13*(4), 165-177.

Bloomfield, P. (2000). Fourier analysis of Time Series, John Wiley & Sons.

Boshnakov, G.N., & Halliday J. (2022). sarima: Simulation and Prediction with Seasonal ARIMA Models, 2022. URL from https://CRAN.R-project.org/package=sarima, Retrieved from https://geobosh.github.io/sarima/ on October 10, 2022.

Cabrero, A., Camba-Mendez, G., Hirsch, A., & Nieto, F. (2009). Modeling the daily banknotes in circulation in the context of the liquidity management of the European Central Bank. *Journal of Forecasting*, *28*, 194-217, https://doi.org/10.1002/for.1118

De Livera, A.M., Hyndman, R.J., & Snyder, R.D. (2011). Forecasting time series with complex seasonal patterns using exponential smoothing. *Journal of American Statistical. Association. 106*(496), 1513-152. Retrieved from https://www.tandfonline.com/doi/abs/10.1198/jasa.2011.tm09771 on October 10, 2022.

Ekinci, Y., Lu, J.C., & Duman, E. (2015). Optimization of ATM cash replenishment with group-demand forecasts. *Expert Systems with Applications*, *42*(7), 3480-3490. Retrieved from http://www.sciencedirect.com/science/article/pii/S0957417414007842 on October 10, 2022.

Fallahtafti, A., Aghaaminiha, M., Akbarghanadian, S., & Weckman, G.R. (2022). Forecasting ATM cash demand before and during the COVID-19 pandemic using an extensive evaluation of statistical and machine learning models. *SN Computer Science*, *3*(2), 1-19. https://doi.org/10.1007/s42979-021-01000-0

Findley, D.F., & Monsell, B.C. (2009). Modeling Stock Trading Day Effects Under Flow Day-of-Week Effect Constraints. *Journal of Official Statistics*, *25*(3), 415-430.

Gos, M., Krzyszczak, J., Baranowski, P., Murat, M., & Malinowska, I. (2020). Combined TBATS and SVM model of minimum and maximum air temperatures applied to wheat yield prediction at different locations in Europe. *Agricultural and Forest Meteorology*, *281*, 107827-41. https://doi.org/10.1016/j.agrformet.2019.107827

Gurgul, G., & Suder, M. (2012). Efekt kalendarza wypłat z bankomatów sieci Euronet. *Zeszyty Naukowe Wyższej Szkoły Ekonomii i Informatyki w Krakowie*, *8*, 25-42.

Gurgul, H., & Suder, M. (2013). Modeling of Withdrawals from Selected ATMs of the "Euronet" Network. *Managerial Economics*, *13,* 65-82. https://doi.org/10.7494/manage.2013.13.65

Gurgul, H., & Suder, M. (2016). Calendar and Seasonal Effects on the Size of withdrawals from ATMs Managed by EURONET. *Statistics in Transition New Series*, *17*(4), 691-722. https://doi.org/10.21307/stattrans-2016-047

Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: principles and practie*, 3rd edition, OTexts: Melbourne, Australia.

Hyndman, R., Koehler, A.B., Ord, J.K., & Snyder, R.D. (2008). Forecasting with Exponential Smoothing. Springer.

Johnson, N.L. (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, *36*, 149-176.

Makridakis, S.G, Wheelwright, S.C., & Hyndman, R.J. (1998). Forecasting: Methods and Applications, John Wiley&Sons.

Munim, Z.H. (2022). State-space TBATS model for container freight rate forecasting with improved accuracy. *Maritime Transport Research, 3*, 1-14. https://doi.org/10.1016/j.martra.2022.100057

Perone, G. (2022). Comparison of ARIMA, ETS, NNAR, TBATS and hybrid models to forecast the second wave of COVID-19 hospitalizations in Italy. *The European Journal of Health Economics*, *23*, 917-940.

https://doi.org/10.1007/s10198-021-01347-4

Rodrigues, P., & Esteves, P. (2010). Calendar effects in daily ATM withdrawals. *Economics Bulletin*, *30*(4), 2587-2597. https://core.ac.uk/download/pdf/6471272.pdf

Rafi, M., Wahab, M.T., Khan, M.B., & Raza, H. (2021). Towards optimal ATM cash replenishment using time series analysis. *Journal of Intelligent & Fuzzy Systems*, *41*(6), 5915-5927, https://doi.org/10.3233/JIFS-201953

Sardar, T., Nadim, S.S., Rana, S., & Chattopadhyay, J. (2020). Assessment of lockdown effect in some states and overall India: A predictive mathematical study on COVID-19 outbreak. *Chaos Solitons Fractals*. *139*, 110078.

https://doi.org/ 10.1016/j.chaos.2020.110078

Serengil, S.I., & Ozpinar, A. (2019). ATM cash flow prediction and replenishment optimization with ANN. *International Journal of Engineering Research and Development*, *11*(1), 402-408. https://doi.org/10.29137/umagd.484670

Shumway, R.H., & Stoffer, D.S. (2010). Time Series Analysis and Its Applications. Springer.

Sieja, M., & Wach, K. (2019). The Use of Evolutionary Algorithms for Optimization in the Modern Entrepreneurial Economy: Interdisciplinary Perspective. *Entrepreneurial Business and Economics Review*, *7*(4), 117-130. https://doi.org/10.15678/EBER.2019.070407

Simutis, R., Dilijonas, D., Bastina, L., Friman, J., & Drobinov, P. (2007). Optimization of Cash Management for ATM Network. *Information Technology and Control*, *36(*1A), 117-121.

Simutis, R., Dilijonas, D., & Bastina, L. (2008). Cash demand forecasting for ATM using Neural Networks and support vector regression algorithms. *20th International Conference, EURO Mini Conference, Continuous Optimization and Knowledge-Based Technologies* (EurOPT- 2008), Selected Papers, Vilnius May 20-23, 416-421.

Suder, M. (2015). Zarządzanie siecią bankomatów: wybrane problemy i zagadnienia. *Zeszyty Naukowe Wyższej Szkoły Ekonomii i Informatyki w Krakowie, 11*, 171-193.

Takala, K., Viren, M. (2007). Impact of ATMs on the Use of Cash. *Communications and Strategies*, *66*, 47-61.

Talkhi, N., Fatemi, N.A., Ataei, Z., & Nooghabi, M.J. (2021). Modeling and forecasting number of confirmed and death caused COVID-19 in IRAN: a comparison of time series forecasting methods. *Biomedical Signal Processing and* .Control 66, 102494. https://doi.org/10.1016/j.bspc.2021.102494

Toit, D.J. (2011). ATM Cash Management for South African Retail Bank, Dissertation, 2011, Retrieved from https://www03.core.ac.uk/download/pdf/37345108.pdf on October 10, 2022.

Wagner, M. (2010). Forecasting Daily Demand in Cash Supply Chains. *American Journal of Economics and Business Administration*, *2*(4), 377-383. Retrieved from https://www.thescipub.com/abstract/10.3844/ajebasp.2010.377.383 on October 10, 2022.

Zandevakili, M., & Javanmard, M. (2014). Using fuzzy logic (type II) in the intelligent ATMs' cash management. *International Research Journal of Applied and Basic Sciences*, *8*(10), 1516-1519. (PDF) Using fuzzy logic (type II) in the intelligent ATMs' cash management (researchgate.net). Retrieved on October 10, 2022.

Appendix A1:

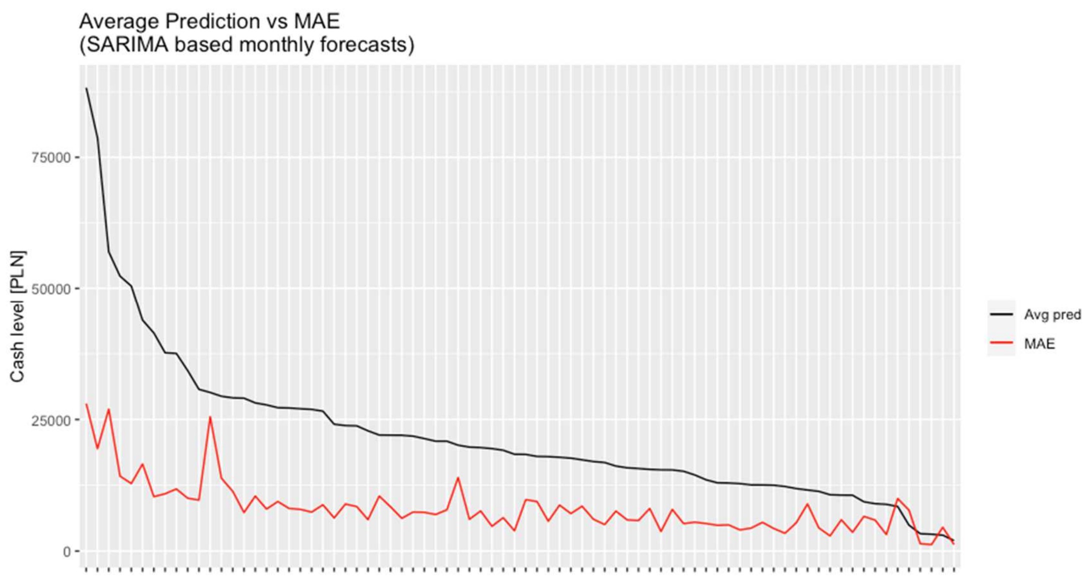
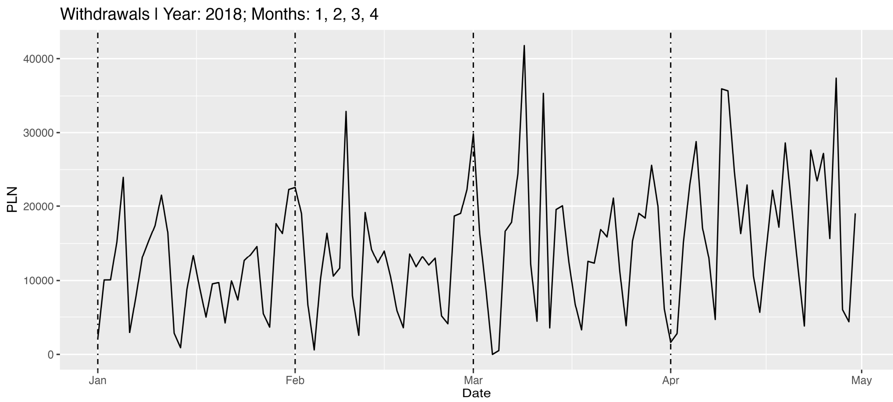


Figure A1. Average levels of monthly cash withdrawals with respective levels of SARIMA-based MAE for 74 ATMs tested

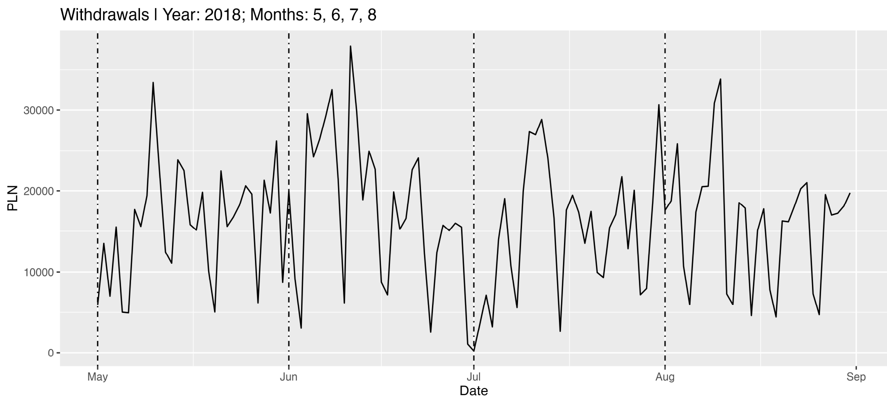
Source: own elaboration.

Appendix A2:

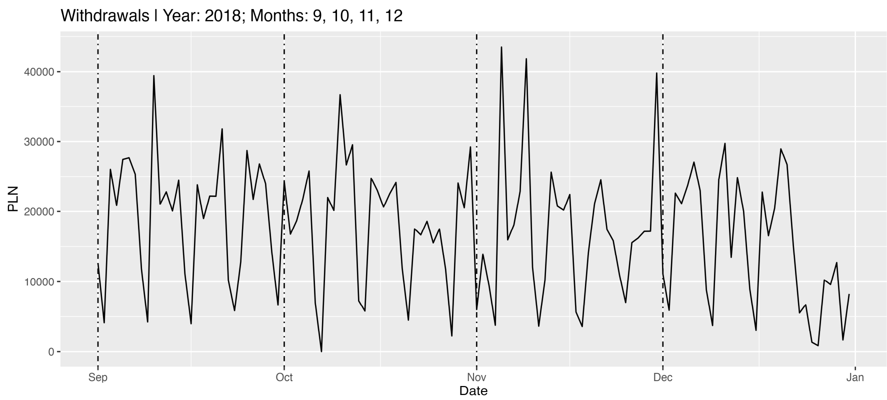
Panel **A**



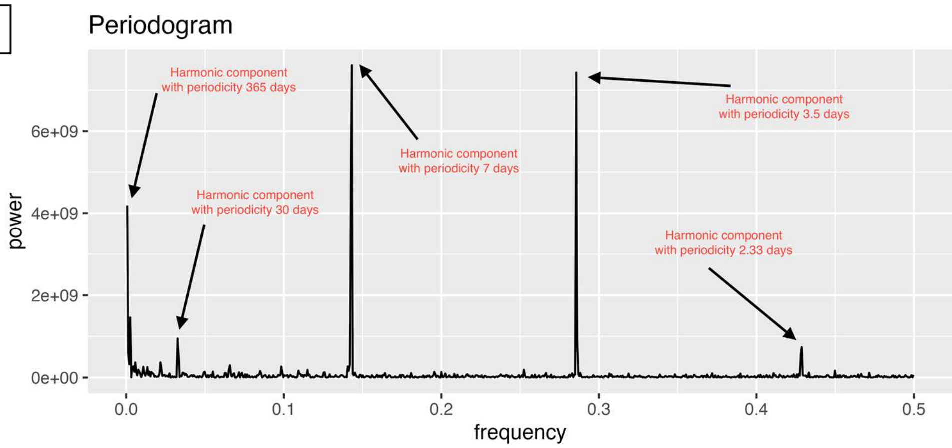
Panel **B**



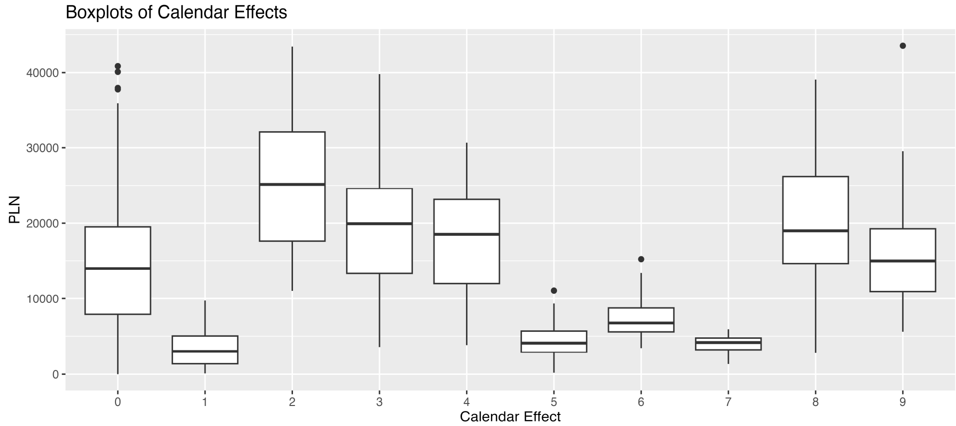
Panel **C**



Panel **D**



Panel **E**



Panel **F**

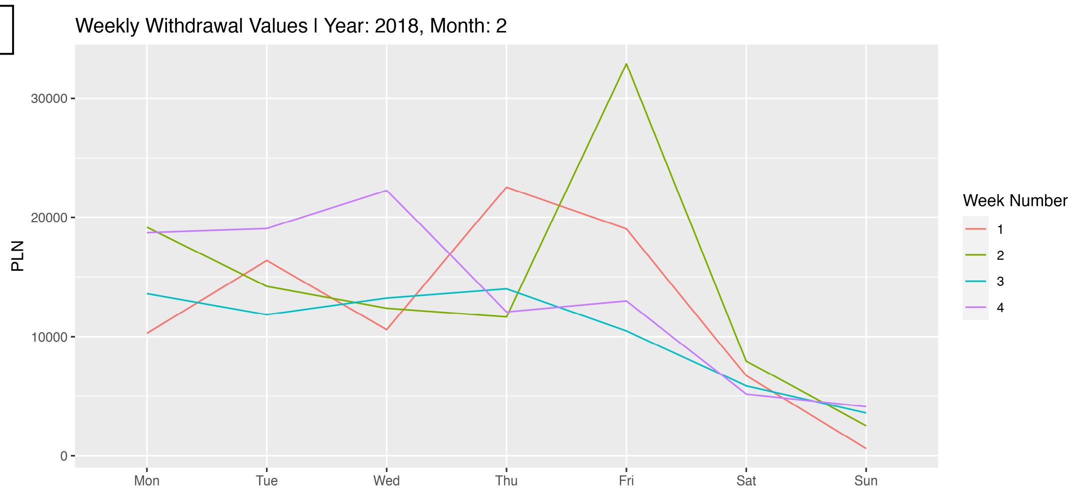


Figure A2. Main features of time series of withdrawals from a randomly selected ATM

Notes: The Figure presents all the effects discussed in detail in Figure 1 but this time derived for a single randomly selected ATM.

Source: own elaboration.

Appendix A3:

The R scripts used in this study may be downloaded from:

https://onedrive.live.com/?v=validatepermission&id=7A073B0828CAC72F%21252218&challengeToken=AI2FhJ7R7MeAoVo

(password: agharticle1321)

Ministry of Education and Science
Republic of Poland

The journal is co-financed in the years 2022-2024 by the Ministry of Education and Science of the Republic of Poland in the framework of the ministerial programme "Development of Scientific Journals" (RCN) on the basis of contract no. RCN/SP/0583/2021/1 concluded on 13 October 2022 and being in force until 13 October 2024.

Authors

The contribution share of the authors is equal and amounts to 25% for each. HG, MS, KS, ŁL – conceptualisation, discussion; HG, MS – literature writing; MS, KS, ŁL – methodology, calculations.

Henryk Gurgul

AGH University of Krakow, Faculty of Management, Full Professor of Economics. His research interests include the application of mathematics in economics, in particular input-output models and financial econometrics. Gurgul is an active member of several international scientific societies, including the International Statistical Institute (elected member).

Correspondence to: Akademia Górniczo-Hutnicza, Wydział Zarządzania, ul. Gramatyka 10, 30-067 Krakow, Poland, e-mail: henryk.gurgul@gmail.com

ORCID http://orcid.org/0000-0002-6192-2995

Łukasz Lach

AGH University of Krakow, Faculty of Management, Associate Professor in Economics and Finance. His research interests cover the application of mathematics in economics, in particular input-output models and energy economics.

Correspondence to: Akademia Górniczo-Hutnicza, Wydział Zarządzania, ul. Gramatyka 10, 30-067 Krakow, Poland, e-mail: llach@agh.edu.pl

ORCID http://orcid.org/0000-0001-8599-1162

Marcin Suder

AGH University of Krakow, Faculty of Management; PhD in Economics and Management (2015, AGH University of Science and Technology, Krakow, Poland). His research interests include organizational management, organizational entrepreneurship, application of mathematics in financial and quantitative research methods.

Correspondence to: Akademia Górniczo-Hutnicza, Wydział Zarządzania, ul. Gramatyka 10, 30-067 Krakow, Poland, e-mail: msuder@agh.edu.pl

ORCID http://orcid.org/0000-0001-6279-7359

Karol Szpyt

AGH University of Krakow. Data Analyst. Masters's degree in IT and Econometrics. His research interests are issues of analyzing large datasets using various technologies.

Correspondence to: Akademia Górniczo-Hutnicza, Wydział Zarządzania, ul. Gramatyka 10, 30-067 Krakow, Poland, e-mail: karolo.szpyt@gmail.com

ORCID http://orcid.org/0000-0002-9983-910X

Acknowledgements and Financial Disclosure

The authors would like to kindly thank the Editors of the journal and two anonymous Referees for their valuable comments on an earlier version of the article.

This study was financed from the statutory funds for the maintenance and development of the research capacity of the Faculty of Management of AGH University of Krakow, Poland.

Conflict of Interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Copyright and License

Published by Krakow University of Economics – Krakow, Poland